

Yangians: their Representations and Characters

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Introduction

The Yangian $Y(\mathfrak{g})$ associated to a finite-dimensional complex simple Lie algebra \mathfrak{g} is a Hopf algebra deformation of the universal enveloping algebra of the Lie algebra $\mathfrak{g}[u]$ of polynomial maps $\mathbb{C} \rightarrow \mathfrak{g}$ (with Lie bracket defined pointwise). It is important to understand the finite-dimensional representations of $Y(\mathfrak{g})$. For one thing, they are closely related to rational solutions of the so-called quantum Yang–Baxter equation (see [7] and [5], Chapter 12). For another, they have recently been shown to act as ‘quantum symmetry groups’ of certain integrable systems (see [1]). In [8], Drinfel’d gave a classification of the finite-dimensional irreducible representations of $Y(\mathfrak{g})$ analogous to the classification of the finite-dimensional irreducible representations of \mathfrak{g} in terms of highest weights, but this left open the problem of giving an ‘explicit’ realization of these representations. This was! solved in [3] when $\mathfrak{g} = \mathfrak{sl}_2$

. We showed there that every finite-dimensional irreducible representation of $Y(\mathfrak{sl}_2)$ is a tensor product of representations which are irreducible under \mathfrak{sl}_2 , and described the action of $Y(\mathfrak{sl}_2)$ on these latter representations completely (for any \mathfrak{g} , $Y(\mathfrak{g})$ contains $U(\mathfrak{g})$ as a (Hopf) subalgebra, so any representation of $Y(\mathfrak{g})$ can be regarded as a representation of \mathfrak{g}). Although it is known that, when $\text{rank}(\mathfrak{g}) > 1$, it is not true that every finite-dimensional irreducible representation of $Y(\mathfrak{g})$ is a tensor product of representations which are irreducible under \mathfrak{g} (cf. [6]), one can hope to identify a larger, but still manageable, class of representations of $Y(\mathfrak{g})$ of which every finite-dimensional irreducible representation of $Y(\mathfrak{sl}_2)$ is a tensor product. Unfortunately, the proof of the tensor product theorem given in [3] depends on the computation of a certain ‘quantum Vandermonde determinant’, and ! this does not seem to generalise e

asily to the higher rank situation. In the first part of this paper (Section 3), we give a proof of the tensor product theorem in the \mathfrak{sl}_2 case using techniques which, as far as possible, admit generalisation to the $\text{rank} > 1$ case; in particular, we make no use of the quantum Vandermonde determinant.

A second approach to understanding the finite-dimensional irreducible representations of $Y(\mathfrak{sl}_2)$ would be to obtain an analogue of the Weyl character formula for them, i.e. a formula for the character of such a representation which depends only on its highest weight (the appropriate notion of character was introduced, and its basic properties obtained, in [9]). In the second part of this paper (Section 4), we give several such formulas in the \mathfrak{sl}_2 case. Their proof depends of the tensor product theorem described above, and thus does not generalise to the higher rank case. Nevertheless, one can hope that the form taken by the formulas in the \mathfrak{sl}_2 case might suggest generalisations for arbitrary \mathfrak{g} , which could then be proved by other methods.

1. Yangians

We take the usual basis $\{H, X^+, X^-\}$ of the Lie algebra \mathfrak{sl}_2 (over \mathbb{C}), so that

$$[H, X^\pm] = \pm 2X^\pm, \quad [X^+, X^-] = H.$$

Let $(\ , \)$ be the invariant symmetric bilinear form on \mathfrak{sl}_2 such that

$$(H, H) = 2, \quad (X^+, X^-) = 1,$$

and denote by Ω the Casimir element

$$\Omega = \sum_{\lambda} I_{\lambda} \otimes I_{\lambda},$$

where $\{I_{\lambda}\}$ is any orthonormal basis of \mathfrak{sl}_2 . We also denote by Ω the element

$$\Omega = \sum_{\lambda} I_{\lambda}^2$$

in the universal enveloping algebra $U(\mathfrak{sl}_2)$.

Definition 1.1. *The Yangian $Y(\mathfrak{sl}_2)$ is the algebra over \mathbb{C} generated by elements $x, J(x)$, for $x \in \mathfrak{sl}_2$, with the following defining relations:*

- (1) $[x, y] \text{ (in } Y(\mathfrak{sl}_2)) = [x, y] \text{ (in } \mathfrak{g}),$
- (2) $J(ax + by) = aJ(x) + bJ(y),$
- (3) $[x, J(y)] = J([x, y]),$
- (4) $[J(x), J([y, z])] + [J(z), J([x, y])] + [J(y), J([z, x])] =$
 $\sum_{\lambda, \mu, \nu} ([x, I_{\lambda}], [[y, I_{\mu}], [z, I_{\nu}]]) \{I_{\lambda}, I_{\mu}, I_{\nu}\},$
- (5) $[[J(x), J(y)], [z, J(w)]] + [[J(z), J(w)], [x, J(y)]] =$
 $\sum_{\lambda, \mu, \nu} ([x, I_{\lambda}], [[y, I_{\mu}], [[z, w], I_{\nu}]]) \{I_{\lambda}, I_{\mu}, J(I_{\nu})\},$

for all $x, y, z \in \mathfrak{sl}_2$, $a, b \in \mathbb{C}$. Here, for any elements $z_1, z_2, z_3 \in Y(\mathfrak{sl}_2)$, we set

$$\{z_1, z_2, z_3\} = \frac{1}{24} \sum_{\pi} z_{\pi(1)} z_{\pi(2)} z_{\pi(3)},$$

the sum being over all permutations π of $\{1, 2, 3\}$.

The Yangian $Y(\mathfrak{sl}_2)$ has a Hopf algebra structure with counit ϵ , comultiplication Δ and antipode S given by

- (6) $\Delta(x) = x \otimes 1 + 1 \otimes x,$
- (7) $\Delta(J(x)) = J(x) \otimes 1 + 1 \otimes J(x) + \frac{1}{2}[x \otimes 1, \Omega],$
- (8) $S(x) = -x, \quad S(J(x)) = -J(x) + x,$
- (9) $\epsilon(x) = \epsilon(J(x)) = 0.$

We shall also need the following presentation of $Y(\mathfrak{sl}_2)$ given in [8]:

Theorem 1.2. *The Yangian $Y(\mathfrak{sl}_2)$ is isomorphic to the associative algebra with generators X_k^\pm , H_k , $k \in \mathbb{N}$, and the following defining relations:*

$$\begin{aligned} (10) \quad & [H_k, H_l] = 0, \quad [H_0, X_k^\pm] = \pm 2X_k^\pm, \quad [X_k^+, X_l^-] = H_{k+l}, \\ (11) \quad & [H_{k+1}, X_l^\pm] - [H_k, X_{l+1}^\pm] = \pm(H_k X_l^\pm + X_l^\pm H_k), \\ (12) \quad & [X_{k+1}^\pm, X_l^\pm] - [X_k^\pm, X_{l+1}^\pm] = \pm(X_k^\pm X_l^\pm + X_l^\pm X_k^\pm). \end{aligned}$$

The isomorphism f between the two realizations of $Y(\mathfrak{sl}_2)$ is given by

$$\begin{aligned} f(H) &= H_0, \quad f(X^\pm) = X_0^\pm, \\ f(J(H)) &= H_1 + \frac{1}{2}(X_0^+ X_0^- + X_0^- X_0^+ - H_0^2), \\ f(J(X^\pm)) &= X_1^\pm - \frac{1}{4}(X_0^\pm H_0 + H_0 X_0^\pm). \quad \square \end{aligned}$$

The presentation 1.1 of $Y(\mathfrak{sl}_2)$ shows that there is a canonical map $\mathfrak{sl}_2 \rightarrow Y(\mathfrak{sl}_2)$ (it is known that this map is injective). Thus, any $Y(\mathfrak{sl}_2)$ -module may be regarded as an \mathfrak{sl}_2 -module.

We shall make use of the following automorphism of $Y(\mathfrak{sl}_2)$.

Proposition 1.3. *There is a one-parameter group $\{\tau_a\}_{a \in \mathbb{C}}$ of Hopf algebra automorphisms of $Y(\mathfrak{sl}_2)$ given in terms of the presentation 1.1 by*

$$\tau_a(x) = x, \quad \tau_a(J(x)) = J(x) + ax,$$

for $x \in \mathfrak{sl}_2$, and in terms of the presentation 1.2 by

$$\tau_a(H_k) = \sum_{r=0}^k \binom{k}{r} a^{k-r} H_r, \quad \tau_a(X_k^\pm) = \sum_{r=0}^k \binom{k}{r} a^{k-r} X_r^\pm. \quad \square$$

This is easily checked using 1.1 and 1.2.

We shall also need the following weak version of the Poincaré–Birkhoff–Witt theorem for $Y(\mathfrak{sl}_2)$.

Proposition 1.4. *Let Y^+ , Y^- and Y^0 be the subalgebras of $Y(\mathfrak{sl}_2)$ generated by the X_k^+ , the X_k^- and the H_k , respectively ($k \in \mathbb{N}$). Then,*

$$Y(\mathfrak{sl}_2) = Y^- \cdot Y^0 \cdot Y^+. \quad \square$$

The proof is straightforward.

2. Representations

If W is an \mathfrak{sl}_2 -module, a non-zero vector $w \in W$ is said to be of weight r if $H.w = rw$, and is said to be an \mathfrak{sl}_2 -highest weight vector if, in addition, $X^+.w = 0$; if $W = U(\mathfrak{sl}_2).w$, then W is called a highest weight \mathfrak{sl}_2 -module with highest weight r . Lowest weight vectors and \mathfrak{sl}_2 -modules are defined similarly. For any $r \in \mathbb{N}$

denote by W_r the unique irreducible highest weight \mathfrak{sl}_2 -module with highest weight r .

Suppose now that V is a $Y(\mathfrak{sl}_2)$ -module. A non-zero vector $v \in V$ is called a $Y(\mathfrak{sl}_2)$ -highest weight vector if v is an eigenvector of H_k , say

$$H_k.v = d_k v, \quad (d_k \in \mathbb{C})$$

and is annihilated by X_k^+ , for all $k \in \mathbb{N}$. Then, V is called a highest weight $Y(\mathfrak{sl}_2)$ -module if $V = Y(\mathfrak{sl}_2).v$ for some $Y(\mathfrak{sl}_2)$ -highest weight vector $v \in V$, and the \mathbb{N} -tuple of scalars $\mathbf{d} = \{d_k\}_{k \in \mathbb{N}}$ is called its highest weight. It is not difficult to show that, for every $\mathbf{d} \in \mathbb{C}^{\mathbb{N}}$, there is an irreducible $Y(\mathfrak{sl}_2)$ -module $V(\mathbf{d})$, unique up to isomorphism, such that $V(\mathbf{d})$ has highest weight \mathbf{d} . Lowest weight vectors and modules for $Y(\mathfrak{sl}_2)$ are defined similarly.

The following theorem of Drinfel'd [8] classifies the finite-dimensional irreducible $Y(\mathfrak{sl}_2)$ -modules. Let \mathcal{P} be the set of monic polynomials in $\mathbb{C}[u]$, where u is an indeterminate.

Theorem 2.1. (i) *Every finite-dimensional irreducible $Y(\mathfrak{sl}_2)$ -module is both highest weight and lowest weight.*

(ii) *If $\mathbf{d} = \{d_k\}_{k \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$, the $Y(\mathfrak{sl}_2)$ -module $V(\mathbf{d})$ is finite-dimensional if and only if there exists $P \in \mathcal{P}$ such that*

$$(13) \quad \frac{P(u+1)}{P(u)} = 1 + \sum_{k=0}^{\infty} d_k u^{-k-1},$$

in the sense that the right-hand side is the Laurent expansion of the left-hand side about $u = \infty$. \square

If V is a finite-dimensional irreducible $Y(\mathfrak{sl}_2)$ -module, we call the associated polynomial P the Drinfel'd polynomial of V .

More generally, if V is any finite-dimensional $Y(\mathfrak{sl}_2)$ -module and $v \in V$ is a $Y(\mathfrak{sl}_2)$ -highest weight vector, with

$$H_k.v = d_k v$$

for some $d_k \in \mathbb{C}$ and all $k \in \mathbb{N}$, it follows from 2.1 that there exists $P \in \mathcal{P}$ such that

$$\frac{P(u+1)}{P(u)} = 1 + \sum_{k=0}^{\infty} d_k u^{-k-1}.$$

Proposition 2.2. *Let V_1, V_2 be finite-dimensional $Y(\mathfrak{sl}_2)$ -modules, and let $v_1 \in V_1, v_2 \in V_2$ be $Y(\mathfrak{sl}_2)$ -highest weight vectors with associated polynomials P_1 and P_2 . Then, $v_1 \otimes v_2$ is a $Y(\mathfrak{sl}_2)$ -highest weight vector in $V_1 \otimes V_2$ with associated polynomial $P_1 P_2$. \square*

This is Proposition 4.6 in [3].

Given a finite-dimensional $Y(\mathfrak{sl}_2)$ -module V , we can define the following associated $Y(\mathfrak{sl}_2)$ -modules:

- (i) $V(a)$: this is obtained pulling back V through τ_a ;
- (ii) the left dual ${}^t V$ and right dual V^t : these are given by the following actions of $Y(\mathfrak{sl}_2)$ on the vector space dual of V :

$$\begin{aligned} (y.f)(v) &= f(S(y).v), & y \in Y(\mathfrak{sl}_2), f \in {}^t V, v \in V, \\ (y.f)(v) &= f(S^{-1}(y).v), & y \in Y(\mathfrak{sl}_2), f \in V^t, v \in V. \end{aligned}$$

Clearly, if V is irreducible, so are all the representations defined above.

Proposition 2.3. *Let U , V and W be finite-dimensional $Y(\mathfrak{sl}_2)$ -modules, and let $a \in \mathbb{C}$. Then,*

- (i) $\text{Hom}_{Y(\mathfrak{sl}_2)}(U, V \otimes W) \cong \text{Hom}_{Y(\mathfrak{sl}_2)}({}^tV \otimes U, W);$
- (ii) $\text{Hom}_{Y(\mathfrak{sl}_2)}(U, W \otimes V) \cong \text{Hom}_{Y(\mathfrak{sl}_2)}(U \otimes V^t, W).$ \square

The proof is straightforward, using the fact that S is a coalgebra anti-automorphism of $Y(\mathfrak{sl}_2)$.

The following result describes the Drinfel'd polynomials of the modules defined above. See [4] for the proof.

Proposition 2.4. *Let V be a finite-dimensional irreducible $Y(\mathfrak{sl}_2)$ -module with Drinfel'd polynomial P , and let $a \in \mathbb{C}$. Then:*

- (i) *The Drinfel'd polynomial of $V(a)$ is $P(u - a)$.*
- (ii) *The Drinfel'd polynomials of tV and V^t are $P(u + 1)$ and $P(u - 1)$, respectively.* \square

Corollary 2.5. ${}^tV \cong V(-1)$ and $V^t \cong V(1)$. \square

We conclude this section with the following result.

Proposition 2.6. *Let V be a finite-dimensional $Y(\mathfrak{sl}_2)$ -module. Then, V is irreducible if and only if V and tV (resp. V and V^t) are both highest weight $Y(\mathfrak{sl}_2)$ -modules.*

Proof. The ‘only if’ part follows from 2.1 (i). For the converse, suppose that V and tV are highest weight (the other case is identical). Let v be a $Y(\mathfrak{sl}_2)$ -highest weight vector in V of weight n (say) for \mathfrak{sl}_2 . Let $0 \neq W$ be an irreducible $Y(\mathfrak{sl}_2)$ -submodule of V , and let m (say) be the highest weight of W as an \mathfrak{sl}_2 -module; thus, $m \leq n$. Then, tW is a quotient of tV , and these \mathfrak{sl}_2 -modules have maximal weights m and n , respectively. Since tV is a $Y(\mathfrak{sl}_2)$ -highest weight module, its highest weight vector must map to a non-zero element of tW . Hence, $n \leq m$. Thus, $m = n$ and $W = V$. \square

3. Classification

For any $a \in \mathbb{C}$, $r \in \mathbb{N}$, the *string* $S_r(a)$ is the set of complex numbers

$$S_r(a) = \{a, a+1, a+2, \dots, a+r-1\}.$$

(Note that this is different from the notation used in [3].) We say that $S_r(a)$ begins at a and ends at $a+r-1$; its cardinality $|S_r(a)| = r$ is often called the length of $S_r(a)$.

Two strings S and T are said to be in special position if $S \cup T$ is a string which is strictly longer than both S and T ; otherwise, S and T are in general position. We leave it to the reader to prove

Lemma 3.1. *Let $a, b \in \mathbb{C}$, $p, q \in \mathbb{N}$. Then, $S_p(a)$ and $S_q(b)$ are in general position if and only if*

- (i) $S_p(a) \subseteq S_q(b)$ or $S_q(b) \subseteq S_p(a)$, or
- (ii) $S_p(a) \cap S_q(b) = S_p(a+1) \cap S_q(b) = S_p(a-1) \cap S_q(b) = \emptyset$. \square

The following result contains the combinatorial properties of strings that we shall need. Again, we leave the proof to the reader.

Proposition 3.2. *Let S_1, S_2, \dots, S_p ($p \geq 1$) be a collection of strings, every pair of which are in general position.*

- (i) *Let $a \in \mathbb{C}$ be such that, for all $n \geq 0$, $a+n \notin \bigcup_{q=1}^p S_q$. Then,*
 - (a) *if $a-1 \notin \bigcup_{q=1}^p S_q$, the string $\{a\}$ is in general position with respect to every S_q ;*
 - (b) *if $a-1 \in S_r$, and if $|S_r|$ is maximal with this property, then $S_r \cup \{a\}$ is a string in general position with respect to every S_q .*
- (ii) *Let $a \in S_t$, and assume that $|S_t|$ is minimal with this property. Then, $S_t \setminus \{a\}$ is a string in general position with respect to every S_q . \square*

A *multiset* is a map $\mathcal{S} \rightarrow \mathbb{N}$, where \mathcal{S} is a finite set of complex numbers. Define the union, intersection and cardinality of multisets in the obvious way.

Corollary 3.3. *Any multiset S can be written uniquely as a union of strings, any two of which are in general position.*

Proof. By induction on the cardinality of S , using 3.2 (ii). \square

We call the decomposition of S into strings given by 3.3 its canonical decomposition.

The following $Y(\mathfrak{sl}_2)$ -modules are the ‘building blocks’ out of which an arbitrary finite-dimensional $Y(\mathfrak{sl}_2)$ -module will be constructed by taking tensor products.

Definition 3.4. *If $a \in \mathbb{C}$, $r \in \mathbb{N}$, then $W_r(a)$ is the finite-dimensional irreducible $Y(\mathfrak{sl}_2)$ -module whose Drinfel’d polynomial is*

$$P_{r,a}(u) = \prod_{b \in S_r(a)} (u - b).$$

The explicit action of the generators H_k, X_k^\pm ($k \in \mathbb{N}$) on a suitable basis of $W_r(a)$ is given in Proposition 2.6 and Corollary 2.7 in [2]. We recall the results:

Proposition 3.5. *For any $r \geq 1$, $a \in \mathbb{C}$, $W_r(a)$ has a basis $\{w_0, w_1, \dots, w_r\}$ on which the action of $Y(\mathfrak{sl}_2)$ is given by*

$$\begin{aligned} X_k^+.w_s &= (s+a)^k(s+1)w_{s+1}, & X_k^-.w_s &= (s+a-1)^k(r-s+1)w_{s-1}, \\ H_k.w_s &= ((s+a-1)^k s(r-s+1) - (s+a)^k(s+1)(r-s))w_s \end{aligned}$$

(it is understood that $w_{-1} = w_{r+1} = 0$). In particular, $W_r(a)$ is isomorphic to W_r as an \mathfrak{sl}_2 -module. \square

Using these formulas, and the formula for the comultiplication of $Y(\mathfrak{sl}_2)$ given in 1.1, it is straightforward to prove

Proposition 3.6. *Let $a \in \mathbb{C}$ and let $v_0 = v^+ \otimes v^- - v^- \otimes v^+$, where v^+ (resp. v^-) is an \mathfrak{sl}_2 -highest (resp. lowest) weight vector in W_1 .*

(i) $W_1(a+1) \otimes W_1(a) = Y(\mathfrak{sl}_2).(v^+ \otimes v^-)$, and we have a short exact sequence of $Y(\mathfrak{sl}_2)$ -modules

$$0 \rightarrow Y(\mathfrak{sl}_2).v_0 \rightarrow W_1(a+1) \otimes W_1(a) \rightarrow W_2(a) \rightarrow 0,$$

where $Y(\mathfrak{sl}_2).v_0$ is the one-dimensional trivial module.

(ii) $W_1(a) \otimes W_1(a+1) = Y(\mathfrak{sl}_2).v_0$ and we have a short exact sequence of $Y(\mathfrak{sl}_2)$ -modules

$$0 \rightarrow Y(\mathfrak{sl}_2).(v^+ \otimes v^-) \rightarrow W_1(a) \otimes W_1(a+1) \rightarrow \mathbb{C} \rightarrow 0,$$

where $Y(\mathfrak{sl}_2).(v^+ \otimes v^-) \cong W_2(a)$.

(iii) ${}^t W_1(a) \cong W_1(a-1)$, $W_1(a)^t \cong W_1(a+1)$. \square

We use these computations to prove

Proposition 3.7. *Let $a_1, a_2, \dots, a_r \in \mathbb{C}$, $r \geq 1$. Then:*

(i) *if $a_j - a_i \neq 1$ when $i < j$,*

$$W_1(a_1) \otimes W_1(a_2) \otimes \cdots \otimes W_1(a_r)$$

is a highest weight $Y(\mathfrak{sl}_2)$ -module;

(ii) *if $a_i - a_j \neq 1$ when $i < j$,*

$$W_1(a_1) \otimes W_1(a_2) \otimes \cdots \otimes W_1(a_r)$$

does not contain a $Y(\mathfrak{sl}_2)$ -highest weight vector of weight $< r$ for \mathfrak{sl}_2 .

Proof. (i) By induction on r . If $r = 1$, there is nothing to prove, and the $r = 2$ case is contained in 3.6. Assume now that $r > 2$ and that the result is known for $r - 1$. If $W_1(a_1) \otimes W_1(a_2) \otimes \cdots \otimes W_1(a_r)$ is not a highest weight $Y(\mathfrak{sl}_2)$ -module, it has an irreducible quotient $V(P)$, say, so that

$$\text{Hom}_{Y(\mathfrak{sl}_2)}(W_1(a_1) \otimes W_1(a_2) \otimes \cdots \otimes W_1(a_r), V(P)) \neq 0.$$

By 2.3 and 3.6,

$$\text{Hom}_{Y(\mathfrak{sl}_2)}(W_1(a_1) \otimes \cdots \otimes W_1(a_r), W_1(a_r+1) \otimes V(P)) \neq 0$$

Let F be a non-zero element of this space of homomorphisms. By the induction hypothesis,

$$W_1(a_2) \otimes \cdots \otimes W_1(a_r) = Y(\mathfrak{sl}_2).(v^+)^{\otimes r-1},$$

where v^+ denotes an \mathfrak{sl}_2 -highest weight vector in W_1 , so $F((v^+)^{\otimes r-1})$ must be a non-zero multiple of $v^+ \otimes v_P$, where v_P is a $Y(\mathfrak{sl}_2)$ -highest weight vector in $V(P)$. By 2.2, $a_1 + 1 = a_i$ for some $i \geq 2$, contradicting our assumption.

(ii) Let $0 \neq v \in W_1(a_1) \otimes W_1(a_2) \otimes \cdots \otimes W_1(a_r)$ be a $Y(\mathfrak{sl}_2)$ -highest weight vector of weight s for \mathfrak{sl}_2 , and let $V = Y(\mathfrak{sl}_2).v$. Then,

$$\text{Hom}_{Y(\mathfrak{sl}_2)}(V, W_1(a_1) \otimes W_1(a_2) \otimes \cdots \otimes W_1(a_r)) \neq 0,$$

and so, by 2.3 and 3.6,

$$\text{Hom}_{Y(\mathfrak{sl}_2)}(W_1(a_r - 1) \otimes \cdots \otimes W_1(a_1 - 1), {}^tV) \neq 0.$$

By part (i), $W_1(a_r - 1) \otimes W_1(a_2 - 1) \otimes \cdots \otimes W_1(a_1 - 1)$ is a $Y(\mathfrak{sl}_2)$ -highest weight module, so tV contains a vector of weight r for \mathfrak{sl}_2 . It follows that $r \leq s$, and hence that $r = s$. \square

This result has several consequences.

Corollary 3.8. *Let $P_1, P_2, \dots, P_r \in \mathcal{P}$, $r \geq 1$, and assume that, if a_i is a root of P_i and a_j a root of P_j , where $i < j$, then $a_j - a_i \neq 1$. Then,*

$$V(P_1) \otimes V(P_2) \otimes \cdots \otimes V(P_r)$$

is a highest weight $Y(\mathfrak{sl}_2)$ -module.

Proof. Let $\{a_{1i}, a_{2i}, \dots, a_{d_i i}\}$ be the multiset of roots of P_i , where $d_i = \deg(P_i)$ and each root is repeated according to its multiplicity. Order the roots so that $a_{ti} - a_{si} \neq 1$ if $s < t$. Then, by 2.2, $V(P_s)$ is a quotient of

$$V_i = W_1(a_{1i}) \otimes W_1(a_{2i}) \otimes \cdots \otimes W_1(a_{d_i i}),$$

and by 3.7 (i),

$$V_1 \otimes V_2 \otimes \cdots \otimes V_r$$

is a highest weight $Y(\mathfrak{sl}_2)$ -module. Hence, $V(P_1) \otimes \cdots \otimes V(P_r)$ is a highest weight $Y(\mathfrak{sl}_2)$ -module. \square

Corollary 3.9. *Every finite-dimensional irreducible $Y(\mathfrak{sl}_2)$ -module is (isomorphic to) a quotient (resp. a submodule) of a tensor product of modules of the form $W_1(a)$, for $a \in \mathbb{C}$.*

Proof. If $P \in \mathcal{P}$, let $d = \deg(P)$ and order the roots a_1, a_2, \dots, a_d of P (repeated according to multiplicity) so that $a_j - a_i \neq 1$ if $i < j$. Then, $V(P)$ is a quotient of

$$W_1(a_1) \otimes W_1(a_2) \otimes \cdots \otimes W_1(a_d).$$

The corresponding statement about submodules follows by taking (left or right) duals. \square

To obtain the final consequence of 3.7, we need

Lemma 3.10. *Let V and W be finite-dimensional irreducible $Y(\mathfrak{sl}_2)$ -modules. Then, $V \otimes W$ is irreducible if and only if $V \otimes W$ and $W \otimes V$ are both highest weight $Y(\mathfrak{sl}_2)$ -modules. In particular, $V \otimes W$ is irreducible if and only if $W \otimes V$ is irreducible.*

Proof. Let v^+ and w^+ be $Y(\mathfrak{sl}_2)$ -highest weight vectors in V and W , respectively, and let their Drinfel'd polynomials be P and Q . Assume that $V \otimes W$ is irreducible. The irreducible quotient of the submodule of $W \otimes V$ generated by $w^+ \otimes v^+$ has Drinfel'd polynomial PQ , by 2.2, and hence is isomorphic to $V \otimes W$, by 2.1. For dimensional reasons, the subquotient must therefore be $W \otimes V$. Hence, $W \otimes V$ is irreducible. That both $V \otimes W$ and $W \otimes V$ are highest weight now follows from 2.1.

Conversely, assume that $V \otimes W$ and $W \otimes V$ are highest weight. If $V \otimes W$ is reducible, it contains an irreducible $Y(\mathfrak{sl}_2)$ -submodule Z , say, whose maximal weight as an \mathfrak{sl}_2 -module is strictly less than that of $V \otimes W$. Using 2.3, we get a non-zero homomorphism

$$W^t \otimes V^t \rightarrow Z^t.$$

Using 2.5 and twisting by τ_{-1} , we get a non-zero homomorphism

$$W \otimes V \rightarrow Z.$$

This contradicts the fact that $W \otimes V$ is highest weight. \square

The following result is now immediate from 2.3, 3.8 and 3.10.

Corollary 3.11. *Let $P_1, \dots, P_r \in \mathcal{P}$, $r \geq 1$, and assume that, if a_i is a root of P_i , a_j a root of P_j , and $i < j$, then $1 \neq a_i - a_j \neq -1$. Then,*

$$V(P_1) \otimes \dots \otimes V(P_r)$$

is an irreducible $Y(\mathfrak{sl}_2)$ -module. \square

We are now in a position to take the crucial step towards the classification of the finite-dimensional irreducible $Y(\mathfrak{sl}_2)$ -modules.

Proposition 3.12. *Let $a_1, \dots, a_p \in \mathbb{C}$, $r_1, \dots, r_p \in \mathbb{N}$, $p \geq 1$, and assume that $S_{r_i}(a_i) \subseteq S_{r_1}(a_1)$ for all $i = 1, \dots, p$. Then,*

$$W_{r_1}(a_1) \otimes \dots \otimes W_{r_p}(a_p)$$

is an irreducible $Y(\mathfrak{sl}_2)$ -module.

Proof. By induction on p . If $p = 1$, there is nothing to prove. Now assume that $p = 2$. To simplify the notation in this case, we consider $W_k(a) \otimes W_l(b)$ instead of $W_{r_1}(a_1) \otimes W_{r_2}(a_2)$, and assume that $S_l(b) \subseteq S_k(a)$ (so $l \leq k$).

Suppose first that $l = 1$. If $k = 1$, then $a = b$ and the result follows from 3.11. If $k = 2$, then $b = a$ or $a + 1$. Assume that $b = a + 1$ (the other case is similar). By 3.8, $W_1(a + 1) \otimes W_2(a)$ is a highest weight $Y(\mathfrak{sl}_2)$ -module, since the roots of the Drinfel'd polynomial of $W_2(a)$ are a and $a + 1$. By 3.10, it suffices to prove that $W_2(a) \otimes W_1(a + 1)$ is a highest weight $Y(\mathfrak{sl}_2)$ -module. Assuming otherwise, $W_2(a) \otimes W_1(a + 1)$ has an irreducible quotient $Y(\mathfrak{sl}_2)$ -module, which must have highest weight 1 as an \mathfrak{sl}_2 -module, and hence must be isomorphic as a $Y(\mathfrak{sl}_2)$ -module to $W_1(c)$, for some $c \in \mathbb{C}$. By 2.3 and 2.5, this implies the existence of a non-zero homomorphism of $Y(\mathfrak{sl}_2)$ -modules

$$F : W_2(a) \rightarrow W_1(c) \otimes W_1(a)$$

Since F must, for weight reasons, map the highest weight vector in $W_2(a)$ to a non-zero multiple of the tensor product of the highest weight vectors in $W_1(c)$ and $W_1(a)$, 2.2 implies that $c = a + 1$, contradicting 3.6 (i).

Assume now that $k > 2$ and that the result is known for smaller values of k (we are still assuming that $l = 1$). We consider three cases:

Case I: $b \neq a$ and $b \neq a + k - 1$. Then,

$$S_{k-1}(a+1) \supseteq S_1(b) \subseteq S_{k-1}(a),$$

so by the induction hypothesis on k ,

$$W_{k-1}(a) \otimes W_1(b) \quad \text{and} \quad W_1(b) \otimes W_{k-1}(a+1)$$

are both irreducible $Y(\mathfrak{sl}_2)$ -modules. By 3.8 and the assumption $b \neq a + k - 1$,

$$W_1(a+k-1) \otimes W_{k-1}(a) \otimes W_1(b)$$

is a highest weight $Y(\mathfrak{sl}_2)$ -module, and hence so is its quotient $W_k(a) \otimes W_1(b)$ (note that $W_1(a+k-1) \otimes W_{k-1}(a)$ is highest weight by 3.8). Similarly, by considering

$$W_1(b) \otimes W_{k-1}(a+1) \otimes W_1(a)$$

and using the assumption $b \neq a$, one sees that $W_1(b) \otimes W_k(a)$ is highest weight. Lemma 3.10 completes the proof.

Case II: $b = a + k - 1$. By the induction hypothesis on k , $W_1(b) \otimes W_{k-1}(a+1)$ is irreducible. On the other hand, since $k > 2$, by 3.7 and 3.10,

$$W_1(b) \otimes W_1(a) \cong W_1(a) \otimes W_1(b)$$

as $Y(\mathfrak{sl}_2)$ -modules, so

$$(14) \quad W_{k-1}(a+1) \otimes W_1(a) \otimes W_1(b) \cong W_{k-1}(a+1) \otimes W_1(b) \otimes W_1(a)$$

as $Y(\mathfrak{sl}_2)$ -modules. By 3.8, the right-hand side of (14) is highest weight, hence so is the left-hand side. Hence, its quotient $W_k(a) \otimes W_1(b)$ is highest weight. That $W_1(b) \otimes W_k(a)$ is highest weight is immediate from 3.8, so 3.10 again completes the proof.

Case III: $b = a$. This is similar to Case II. We omit the details.

We have now proved the result when $l = 1$ (and $p = 2$). We next assume that $l > 1$ and that the result is known for smaller values of l . We prove the result for l by induction on k , starting at $k = l$.

If $k = l$, then $a = b$ and the induction hypothesis on l gives that

$$W_1(a+l-1) \otimes W_{l-1}(a) \otimes W_l(a)$$

is highest weight, and hence so is its quotient $W_l(a) \otimes W_l(a)$. By 3.10, this last module is irreducible.

For the inductive step, we distinguish three cases:

Case I: $b - a \neq 0$ or $k - l$. Then,

$$S_{k-1}(a+1) \supseteq S_l(b) \subseteq S_{k-1}(a).$$

By the induction hypothesis on k ,

$$W_l(b) \otimes W_{k-1}(a+1) \quad \text{and} \quad W_{k-1}(a) \otimes W_l(b)$$

are both irreducible. By 3.8,

$$W_l(b) \otimes W_{k-1}(a+1) \otimes W_1(a) \quad \text{and} \quad W_1(a+1) \otimes W_{k-1}(a) \otimes W_l(b)$$

are both highest weight, and hence so are their quotients

$$W_l(b) \otimes W_k(a) \quad \text{and} \quad W_k(a) \otimes W_l(b).$$

Case II: $b - a = k - l$. By the induction hypothesis on l , $W_{l-1}(b) \otimes W_k(a)$ is irreducible. By 3.8,

$$W_1(b+l-1) \otimes W_{l-1}(b) \otimes W_k(a)$$

is highest weight, hence so is $W_l(b) \otimes W_k(a)$. On the other hand,

$$(15) \quad \begin{aligned} W_1(b+l-1) \otimes W_{l-1}(b) \otimes W_k(a) &\cong W_1(b+l-1) \otimes W_k(a) \otimes W_{l-1}(b) \\ &\cong W_k(a) \otimes W_1(b+l-1) \otimes W_{l-1}(b) \end{aligned}$$

as $Y(\mathfrak{sl}_2)$ -modules: the first isomorphism uses the fact that $S_{l-1}(b)$ and $S_k(a)$ are in general position, the second that $\{b+l-1\}$ and $S_k(a)$ are in general position (and both isomorphisms use the induction hypothesis on l). But we saw above that the first tensor product in (15) is highest weight, hence so is the last, and hence so is its quotient $W_k(a) \otimes W_l(b)$.

Case III: $b = a$. This is similar to Case II.

We have now proved 3.12 in the case $p = 2$. Assume next that $p > 2$ and that the result is known for smaller values of p . Let S be the union of the strings $S_{r_i}(a_i)$, $i = 1, \dots, p$, considered as a set with multiplicities. We prove the result for p by induction on $|S|$. If $|S| = p$, we are considering a tensor product of the form $W_1(a)^{\otimes p}$, which is irreducible by 3.11.

Assume now that $|S| > p$ and that the result is known for smaller values of $|S|$. Note that $a_1 + r_1 - 1 \in S$ but $a_1 + n \notin S$ if $n \geq r_1$. Let $S' = S \setminus \{a_1 + r_1 - 1\}$, and let i be such that $a_1 + r_1 - 1 \in S_{r_i}(a_i)$ and such that r_i is minimal with this property. By 3.2 (ii),

$$S' = \bigcup_{j \neq i} S_{r_j}(a_j) \cup (S_{r_i}(a_i) \setminus \{a_1 + r_1 - 1\})$$

is the canonical decomposition of S' . By the induction hypothesis on $|S|$,

$$(16) \quad W_{r_i-1}(a_i) \otimes \bigotimes W_{r_j}(a_j)$$

is irreducible. (Note that, by 3.10 and the induction hypothesis on p , the second tensor product in (16) is independent of the order of the factors, up to isomorphism.) By 3.8,

$$W_1(a_1 + r_1 - 1) \otimes W_{r_i-1}(a_i) \otimes \bigotimes_{j \neq i} W_{r_j}(a_j)$$

is highest weight, hence so is its quotient

$$W_{r_i}(a_i) \otimes \bigotimes_{j \neq i} W_{r_j}(a_j).$$

By the $p = 2$ case, this module is unchanged, up to isomorphism, by successively interchanging adjacent factors in the tensor product, so we deduce that

$$\bigotimes_{j \neq i} W_{r_j}(a_j) \otimes W_{r_i}(a_i)$$

is also highest weight. The usual application of 3.10 completes the proof. \square

We can now prove the main result of this section.

Theorem 3.13. *Let $P \in \mathcal{P}$, and write the multiset $S(P)$ of roots of P as a union of strings in general position, say*

$$S(P) = \bigcup_{i=1}^p S_{r_i}(a_i).$$

Then, as $Y(\mathfrak{sl}_2)$ -modules,

$$V(P) \cong \bigotimes_{i=1}^p W_{r_i}(a_i)$$

(the factors in the tensor product can be taken in any order).

Proof. By induction on p . If $p = 1$, there is nothing to prove. Assume now that $p > 1$ and that the result is known for smaller values of p .

If all the strings $S_{r_i}(a_i)$, $i = 1, \dots, p$, are contained in a single string, the result was proved in 3.12. Otherwise, let r_i be maximal among r_1, \dots, r_p , and let

$$S' = \bigcup_{\{j | S_{r_j}(a_j) \subseteq S_{r_i}(a_i)\}} S_{r_j}(a_j), \quad S'' = S \setminus S'$$

(with S , S' and S'' considered as sets with multiplicities). Then, $S = S' \cup S''$ and

$$S' \cap S'' = (S' + 1) \cap S'' = (S' - 1) \cap S'' = \emptyset.$$

Let $P', P'' \in \mathcal{P}$ have multisets of roots S' and S'' , respectively. By 3.11, $V(P') \otimes V(P'')$ is irreducible. By the induction hypothesis, $V(P')$ and $V(P'')$ are both isomorphic to tensor products as in the statement of the theorem, so an application of 3.10 to re-order the factors, if necessary, completes the proof. \square

Remark It is instructive to consider the classical analogue of 3.13. As we mentioned in the Introduction, the classical analogue of $V(\mathfrak{sl}_2)$ is $U(\mathfrak{sl}_2[x])$.

Theorem 3.14. *Every finite-dimensional irreducible $\mathfrak{sl}_2[u]$ -module V is generated by a vector v such that*

$$(H \otimes u^k).v = d_k v, \quad (X^+ \otimes u^k).v = 0$$

for some $d_k \in \mathbb{C}$ and all $k \in \mathbb{N}$. Moreover, there exists a monic polynomial $P \in \mathbb{C}[u]$ such that

$$\sum_{k=0}^{\infty} d_k u^{-k-1} = \frac{1}{P(u)} \frac{dP}{du}. \quad \square$$

It is not difficult to prove (cf. [2]) that every finite-dimensional irreducible $\mathfrak{sl}_2[u]$ -module is a tensor product of modules of the form $W_r(a)$, for some $r \geq 1$, $a \in \mathbb{C}$, where $W_r(a)$ is obtained by pulling back W_r via the Lie algebra homomorphism $\mathfrak{sl}_2[u] \rightarrow \mathfrak{sl}_2$ given by setting $u = a$. The polynomial associated to $W_r(a)$ is $(u-a)^r$, and the polynomial is multiplicative on irreducible tensor products (cf. 2.2).

4. Characters

The appropriate definition of the character of a finite-dimensional $Y(\mathfrak{sl}_2)$ -module was given in [9]. Let \mathcal{L} be the subgroup of the group of units of the ring $\mathbb{C}[[u^{-1}]]$ of the form

$$f_{\mathbf{d}}(u) = 1 + \sum_{k=0}^{\infty} d_k u^{-k-1},$$

for some $\mathbf{d} = \{d_k\}_{k \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$, and let $\mathbb{C}[\mathcal{L}]$ be its group algebra. If $f \in \mathcal{L}$, we write $e(f)$ for the corresponding basis element of $\mathbb{C}[\mathcal{L}]$.

If V is a finite-dimensional $Y(\mathfrak{sl}_2)$ -module, and $\mathbf{d} = \{d_k\}_{k \in \mathbb{N}}$, set

$$V_{\mathbf{d}} = \{v \in V \mid (H_k - d_k)^N . v = 0 \text{ for } N \gg 0\},$$

and if $P \in \mathcal{P}$ is such that

$$\frac{P(u+1)}{P(u)} = f_{\mathbf{d}}(u)$$

(cf. 2.1 (ii)), set $e(P) = e(f_{\mathbf{d}})$ (abusing notation a little).

Definition 4.1. *If V is a finite-dimensional $Y(\mathfrak{sl}_2)$ -module, its character is*

$$\text{ch}(V) = \sum_{\mathbf{d}} \dim(V_{\mathbf{d}}) e(f_{\mathbf{d}}).$$

The main result proved in [9] is

Proposition 4.2. (i) *If*

$$0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$$

is a short exact sequence of $Y(\mathfrak{sl}_2)$ -modules, then

$$\text{ch}(V) = \text{ch}(U) + \text{ch}(W)$$

(ii) If V and W are finite-dimensional $Y(\mathfrak{sl}_2)$ -modules,

$$\text{ch}(V \otimes W) = \text{ch}(V)\text{ch}(W). \quad \square$$

Part (i) plus Jordan–Hölder means that it suffices to compute the characters of the irreducible modules $V(P)$ ($P \in \mathcal{P}$). To state the character formula, define, for $r \geq 1$, $a \in \mathbb{C}$,

$$x_a = e\left(\frac{u-a+1}{u-a}\right) = e\left(1 + \sum_{k=0}^{\infty} a^k u^{-k-1}\right) \in \mathbb{C}[\mathcal{L}],$$

$$y_{r,a} = 1 + \sum_{s=1}^r \sum_{t=1}^s x_{a+r-t}^{-1} x_{a+r-t+1}^{-1} \quad (\text{and } y_{r,a} = 1 \text{ if } r \leq 0),$$

$$m_{r,a}(P) = \max\{n \in \mathbb{N} \mid (u-a)^n, (u-a-1)^n, \dots, (u-a-r+1)^n \text{ all divide } P(u)\}.$$

The main result in this section is

Theorem 4.3. *For any $P \in \mathcal{P}$,*

$$(17) \quad \text{ch}(V(P)) = e(P) \prod_{r \geq 1, a \in \mathbb{C}} \left(\frac{y_{r,a} y_{r-2,a+1}}{y_{r-1,a} y_{r-1,a+1}} \right)^{m_{r,a}(P)}.$$

Remarks 1. All but finitely many terms in the product (17) are equal to one, since $m_{r,a}(P) = 0$ unless $r \leq \deg(P)$ and a is a root of P .

2. It is clear that, if $Q \in \mathcal{P}$, we have

$$m_{r,a}(PQ) \geq m_{r,a}(P) + m_{r,a}(Q),$$

but strict inequality may occur (e.g. if $P(u) = u - a$ and $Q(u) = u - a - 1$, then $m_{2,a}(PQ) = 1$ but $m_{2,a}(P) = m_{2,a}(Q) = 0$).

3. The definition of $m_{r,a}$ may be reformulated in terms of the ‘Yangian derivative’:

$$D_Y(P) = P(u+1) - P(u).$$

It is clear that

$$m_{r,a}(P) = \max\{n \in \mathbb{N} \mid (u-a)^n \text{ divides } P, D_Y P, \dots, D_Y^{r-1} P\},$$

i.e. that $m_{r,a}(P)$ is the multiplicity of a as a common root of $P, D_Y P, \dots, D_Y^{r-1} P$.

Before proving 4.2, we note some consequences. Let $\text{res} : \mathcal{L} \rightarrow \mathbb{C}$ be the homomorphism given by

$$\text{res}(f_{\mathbf{d}}) = d_0,$$

and let $\mathcal{L}_{\mathbb{Z}} = \text{res}^{-1}(\mathbb{Z})$. We also denote by res the corresponding algebra homomorphisms $\mathbb{C}[\mathcal{L}] \rightarrow \mathbb{C}[\mathbb{C}]$ and $\mathbb{C}[\mathcal{L}_{\mathbb{Z}}] \rightarrow \mathbb{C}[\mathbb{Z}]$. For any finite-dimensional $Y(\mathfrak{sl}_2)$ -module V , we have $\text{ch}(V) \in \mathbb{C}[\mathcal{L}_{\mathbb{Z}}]$. Indeed, this follows from 4.3 if V is irreducible since, for any $P \in \mathcal{P}$, $r \geq 1$, $a \in \mathbb{C}$,

$$\text{res}(e(P)) = e(\deg(P)), \quad \text{res}(y_{r,a}) = \sum_{s=0}^r e(-2s) = z_r \quad (\text{say}),$$

and the general case follows from 4.2 (i). Now, $\text{res}(\text{ch}(V)) = \text{ch}_{\mathfrak{sl}_2}(V)$, the character of V regarded as an \mathfrak{sl}_2 -module. Noting that

$$\sum_{a \in \mathbb{C}} m_{1,a}(P) = \deg(P)$$

and, for $r > 1$,

$$\sum_{a \in \mathbb{C}} m_{r,a}(P) = \text{total number of strings of length } r \text{ in } P = m_r(P),$$

say, we obtain

Corollary 4.4. *For any $P \in \mathcal{P}$,*

$$\text{ch}_{\mathfrak{sl}_2}(V(P)) = e(\deg(P)) z_1^{\deg(P)} \prod_{r=2}^{\infty} \left(\frac{z_r z_{r-2}}{z_{r-1}^2} \right)^{m_r(P)}. \quad \square$$

Similarly, applying the augmentation homomorphism $\mathbb{C}[\mathcal{L}] \rightarrow \mathbb{C}$ to 4.3 gives

Corollary 4.5. *For any $P \in \mathcal{P}$,*

$$\dim(V(P)) = 2^{\deg(P)} \prod_{r=2}^{\infty} \left(\frac{r^2 - 1}{r^2} \right)^{m_r(P)}. \quad \square$$

It follows from 3.9 that, for any $P \in \mathcal{P}$, $\text{ch}(V(P))$ is an alternating sum of tensor products of the characters

$$\chi_a = \text{ch}(W_1(a)).$$

The next result makes this explicit. By 3.12,

$$(18) \quad \text{ch}(V(P)) = \prod_{r \geq 1, a \in \mathbb{C}} \text{ch}(W_r(a))^{N_{r,a}(P)},$$

where $N_{r,a}(P)$ is the number of strings of length r beginning at a in the canonical decomposition of $S(P)$, so it suffices to consider $P = P_{r,a}$.

Proposition 4.6. *For any $r \geq 1$, $a \in \mathbb{C}$,*

$$\text{ch}(W_r(a)) = \sum_{s=0}^{\lfloor r/2 \rfloor} (-1)^s A_{r,a}^{(s)},$$

where

$$A_{r,a}^{(s)} = \sum \chi_{a+t_1} \chi_{a+t_2} \cdots \chi_{a+t_{r-2s}},$$

the sum being over those integers $t_1, t_2, \dots, t_{r-2s}$ such that

$$r > t_1 > t_2 > \cdots > t_{r-2s} \geq 0 \quad \text{and} \quad t_j \equiv r - j \pmod{2} \text{ for all } j.$$

We shall prove this result after proving the next proposition, which is also the first step in the proof of 4.2.

Proposition 4.7. *For any $r \geq 1$, $a \in \mathbb{C}$,*

$$\text{ch}(W_r(a)) = e(P_{r,a})y_{r,a}.$$

Proof. From 3.6, we read off that the joint eigenvalue of $w_s \in W_r(a)$ is $\mathbf{d}_s = \{d_{k,s}\}_{k \in \mathbb{N}}$, where

$$d_{k,s} = (a + s - 1)^k s(r - s + 1) - (a + s)^k (s + 1)(r - s).$$

This gives

$$f_{\mathbf{d}_s} = \frac{(u - a + 1)(u - a - r)}{(u - a - s + 1)(u - a - s)},$$

so

$$\text{ch}(W_r(a)) = \sum_{s=0}^r e \left(\frac{(u - a + 1)(u - a - r)}{(u - a - s + 1)(u - a - s)} \right).$$

Now,

$$e(P_{r,a}) = e \left(\frac{u - a + 1}{u - a - r + 1} \right),$$

so

$$\text{ch}(W_r(a)) = e(P_{r,a}) \sum_{s=0}^k e \left(\frac{u - a - r}{u - a - r + s} \right) e \left(\frac{u - a - r + 1}{u - a - r + s + 1} \right)$$

(changing the summation index from s to $r - s$). Since

$$e \left(\frac{u - a - r}{u - a - r + s} \right) = \prod_{t=1}^s e \left(\frac{u - a - r + t - 1}{u - a - r + t} \right) = \prod_{t=1}^s x_{a+r-t+1}^{-1},$$

this gives the stated formula. \square

Proof of 4.6. Using 4.7, it is easy to show that

$$\text{ch}(W_{r+2}(a)) = \text{ch}(W_{r+1}(a + 1))\chi_a - \text{ch}(W_r(a + 2)).$$

The formula in 4.6 follows easily from this relation, by using induction on r . \square

If $P \in \mathcal{P}$, let

$$(19) \quad S(P) = \bigcup_{i=1}^p S_{r_i}(a_i)$$

be the canonical decomposition of its multiset of roots $S(P)$. By 3.13 and 4.7, we get

$$\text{ch}(V(P)) = e(P) \prod_{i=1}^p y_{r_i, a_i}.$$

Hence,

$$(20) \quad \text{ch}(V(P)) = e(P) \prod y_{r,a}^{N_{r,a}(P)}.$$

Now let $n_{r,a}(P)$ be the total number of strings of length r beginning at a that are contained in $S(P)$. More precisely, relative to the canonical decomposition (19),

$$n_{r,a}(P) = \sum_{i=1}^p n_i,$$

where

$$n_i = \begin{cases} 1 & \text{if } S_r(a) \subseteq S_{r_i}(a_i), \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that

$$\begin{aligned} n_{r,a}(P) &= N_{r,a}(P) \\ &\quad + N_{r+1,a}(P) + N_{r+1,a-1}(P) \\ &\quad + N_{r+2,a}(P) + N_{r+2,a-1}(P) + N_{r+2,a-2} \\ &\quad + \cdots \\ &= \sum_{s=0}^{\infty} \sum_{t=0}^s N_{r+s,a-t}(P) \end{aligned}$$

(of course, all but finitely many terms in the double sum are zero). These equations are easily inverted to express the N 's in terms of the n 's:

$$N_{r,a}(P) = n_{r,a}(P) - n_{r+1,a}(P) - n_{r+1,a-1}(P) + n_{r+2,a-1}(P).$$

Inserting this into (20), we get

$$\text{ch}(V(P)) = e(P) \prod_{r \geq 1, a \in \mathbb{C}} \left(\frac{y_{r,a} y_{r-2,a+1}}{y_{r-1,a} y_{r-1,a+1}} \right)^{n_{r,a}(P)}.$$

Thus, 4.3 is a consequence of

Proposition 4.8. *For any $m \geq 1$, $a \in \mathbb{C}$, $P \in \mathcal{P}$,*

$$m_{r,a}(P) = n_{r,a}(P).$$

To prove 4.8, we need two lemmas.

Lemma 4.9. *Let $P \in \mathcal{P}$. Then, P has a factorisation*

$$P = P_1 P_2 \cdots P_k,$$

for some $k \geq 1$, such that, if a is a root of P_i and b a root of P_j , then $a - b \in \mathbb{Z}$ if $i = j$ and $a - b \notin \mathbb{Z}$ if $i \neq j$. \square

This is clear

Lemma 4.10. *Let $P \in \mathcal{P}$ be such that every pair of roots of P differ by an integer. Then, P has a factorisation*

$$P = P_1 P_2 \dots P_k$$

such that

- (i) *for all $i = 1, \dots, k$, every string in the canonical decomposition of $S(P_i)$ is a string in the canonical decomposition of $S(P)$;*
- (ii) *$S(P_i) \cap S(P_j) = \emptyset$ if $i \neq j$;*
- (iii) *for all $i = 1, \dots, k$, $S(P_i)$ regarded as a set without multiplicities, is a string.*

Proof. By induction on $\deg(P)$. If $\deg(P) = 0$ or 1 , there is nothing to prove. Let S be a string of maximal length in the canonical decomposition of $S(P)$, and let S_1 be the union, in the sense of sets with multiplicities, of all the strings in the canonical decomposition of $S(P)$ that are contained in S . Let $P_1 \in \mathcal{P}$ be the factor of P such that $S(P_1) = S_1$. By the induction hypothesis,

$$P/P_1 = P_2 \dots P_k,$$

where P_2, \dots, P_k satisfy the conditions of the lemma.

To prove that the factorisation

$$P = P_1 P_2 \dots P_k$$

satisfies the conditions of the lemma, we have to prove that, if $j \neq 1$,

- (a) $S(P_1) \cap S(P_j) = \emptyset$;
- (b) every string in the canonical decomposition of $S(P_1)$ is in general position with respect to every string in the canonical decomposition of $S(P_j)$.

For (a), suppose for a contradiction that there exists $c \in S(P_1) \cap S(P_j)$. Then, $c \in S \cap T$ for some string S in the canonical decomposition of $S(P_1)$ and some string T in the canonical decomposition of $S(P_j)$. But S and T are in general position by construction, so since S has maximal length and $S \cap T \neq \emptyset$, we must have $T \subseteq S$. This contradicts the definition of P_1 .

For (b), suppose for a contradiction that S' is a string in the canonical decomposition of $S(P_1)$, T' a string in the canonical decomposition of $S(P_j)$, where $j \neq 1$, and that S' and T' are in special position. By the argument used in the previous paragraph, $S \cap T' = \emptyset$. This gives two possibilities: either S and S' both begin at some $c \in \mathbb{C}$ and T' ends at $c - 1$, or S and S' both end at some $c' \in \mathbb{C}$ and T' begins at $c' + 1$. In both cases, S and T' are in special position, a contradiction. \square

Proof of 4.8. By induction on the number of strings in the canonical decomposition of $S(P)$. The induction begins with

Case I: $S(P)$ is a string (multiplicities counted). Then,

$$P(u) = (u - b)(u - b - 1) \dots (u - b - k + 1)$$

for some $b \in \mathbb{C}$, $k \geq 1$. In this case, it is easy to see that $m_{r,a}(P)$ and $n_{r,a}(P)$ are both equal to 1 if $r \leq k$ and $a = b, b + 1, \dots$, or $b + k$, and equal to 0 otherwise.

Case II: $S(P)$ is a string (disregarding multiplicities). Let S' be a string of maximal length in the canonical decomposition of $S(P)$ (so that $S(P) = S'$ disregarding

multiplicities), and let S'' be the union of the other strings in the canonical decomposition of $S(P)$. Let $P = P'P''$ be the corresponding factorisation of P . It is clear from the definition of $n_{r,a}$ that

$$n_{r,a}(P) = n_{r,a}(P') + n_{r,a}(P''),$$

and since P' has no repeated roots,

$$m_{r,a}(P) = m_{r,a}(P') + m_{r,a}(P''),$$

so the result follows by induction.

Case III: Any two roots of P differ by an integer. We have a factorisation

$$P = P_1 P_2 \dots P_k,$$

where the P_i satisfy the condition in 4.10. Since $S(P_i) \cap S(P_j) = \emptyset$ if $i \neq j$, it is clear that, for each $r \geq 1$, $a \in \mathbb{C}$, $n_{r,a}(P_i) > 0$ for at most one i , say $i = 1$ without loss of generality, and that $n_{r,a}(P) = n_{r,a}(P_1)$. Hence,

$$n_{r,a}(P) = \sum_{i=1}^k n_{r,a}(P_i).$$

On the other hand, conditions (i) and (ii) in 4.10 imply that, if $a \in S(P_i)$, $b \in S(P_j)$, and $i \neq j$, then $|a - b| \geq 2$. We claim that this implies that

$$(21) \quad m_{r,a}(P) = \sum_{i=1}^k m_{r,a}(P_i),$$

so that Case III follows from Case II.

Suppose that $(u - a)^m, (u - a - 1)^m, \dots, (u - a - r + 1)^m$ all divide P . Since $S(P_i) \cap S(P_j) = \emptyset$ if $i \neq j$, $(u - a)^m$ divides P_j for some j . Similarly, $(u - a - 1)^m$ divides P_i for some i , and we must have $i = j$ otherwise a root of P_j would differ from a root of P_i by less than 2. Continuing in this way, we see that $(u - a)^m, (u - a - 1)^m, \dots, (u - a - r + 1)^m$ all divide P_j and divide no other P_i . This proves that $m_{r,a}(P_i) = 0$ if $i \neq j$, and $m_{r,a}(P) \leq m_{r,a}(P_j)$. Hence,

$$m_{r,a}(P) \leq \sum_{i=1}^k m_{r,a}(P_i).$$

The opposite inequality is obvious (see Remark 2 following 4.3), so (21) is proved.

Case IV: General case. By 4.9, we have a factorisation

$$P = P_1 P_2 \dots P_k,$$

where each P_i satisfies the hypotheses of Case III and, if $i \neq j$, each root of P_i differs by a non-integer from each root of P_j . It is now clear that

$$m_{r,a}(P) = \sum_{i=1}^k m_{r,a}(P_i), \quad n_{r,a}(P) = \sum_{i=1}^k n_{r,a}(P_i),$$

(at most one term in each sum being non-zero), so the result follows from Case III. \square

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